Modelli 1 @ Clamfim

Massimi e minimi vincolati

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Remark

In n variables a critical point $\boldsymbol{x}_0$ is a local minimum for $f \in \mathcal{C}^2$ if for each $k = 1, \ldots, n$

$$\det[H_k(f)(\boldsymbol{x}_0)] > 0$$

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Theorem (Lagrange multipliers) Let $m < n$, V be open in $\mathbb{R}^n$ and f, $g_j : V \to \mathbb{R}$ be $\mathcal{C}^1$ on V for $j = 1, 2, \ldots, m$. Suppose that rank of

$$\frac{\partial(g_1, \ldots g_m)}{\partial(x_1, \ldots, x_n)}$$

is m in $\boldsymbol{x}_0 \in V$ where $g_j(\boldsymbol{x}_0) = 0$ for $j = 1, 2, \ldots, m$ and suppose that $\boldsymbol{x}_0$ is a local extremum for f in the set

$$M = \{\boldsymbol{x} \in V : \ g_j(\boldsymbol{x}) = 0\}.$$

Then there exist scalars $\lambda_1, \ldots, \lambda_m$ such that

$$\nabla \left(f(\boldsymbol{x}_0) - \sum_{k=1}^{m} \lambda_k g(\boldsymbol{x}_0) \right) = \mathbf{0}$$

Sufficient conditions: Bordered Hessian

Find max or min of $f(x, y)$ under the constraint $g(x, y) = 0$

Lagrangean $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

After solving the system

$$\begin{cases} f_x'(x, y) - \lambda g_x'(x, y) = 0, \\ f_y'(x, y) - \lambda g_y'(x, y) = 0, \\ g(x, y) = 0. \end{cases}$$

evaluate

$$\Lambda = \det \begin{bmatrix} L''_{xx} & L''_{xy} & g_x \\ L''_{xy} & L''_{yy} & g_y \\ g_x & g_y & 0 \end{bmatrix}$$

$\Lambda > 0$ maximum

$\Lambda < 0$ minimum

Cobb Douglas

$$f(x,y) = x^a y^{1-a} \to \max$$

$$\text{sub} \quad px + qy - c = 0$$

Assumptions $0 < a < 1$, p, q, $c > 0$.

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Lagrangean $L(x,y;m) = f(x,y) - mw(x,y)$ where $w(x,y) = px + qy - c$. Critical point equations

$$L_x(x,y;m) = ax^{a-1}y^{1-a} - mp = 0 \tag{1a}$$
$$L_y(x,y;m) = (1-a)x^a y^{-a} - mq = 0 \tag{1b}$$
$$L_m(x,y;m) = px + qy - c = 0 \tag{1c}$$

Eliminating m between (1a) and (1b) we get

$$(1-a)px - aqy = 0 \tag{2a}$$

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Solving:

$$\begin{cases} x = \dfrac{ac}{p} \\ y = \dfrac{c(1-a)}{q} \\ m = (1-a)^{1-a} a^a p^{-a} q^{a-1} \end{cases}$$

The critical point is a maximum, in fact the bordered hessian is

$$\begin{bmatrix} (a-1)a\left(\frac{ac}{p}\right)^{a-2}\left(\frac{c-ac}{q}\right)^{1-a} & (1-a)a\left(\frac{ac}{p}\right)^{a-1}\left(\frac{c-ac}{q}\right)^{-a} & p \\ (1-a)a\left(\frac{ac}{p}\right)^{a-1}\left(\frac{c-ac}{q}\right)^{-a} & -\frac{a\left(\frac{ac}{p}\right)^{a}\left(\frac{c-ac}{q}\right)^{-a}q}{c} & q \\ p & q & 0 \end{bmatrix}$$

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so that

$$\det = \frac{a^{a-1}p^{2-a}q^{a+1}}{c(1-a)^a} > 0$$

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Critical point equations

$$\begin{cases} 2 - \dfrac{my^{3/4}}{4x^{3/4}} = 0 \\ 1 - \dfrac{3mx^{1/4}}{4y^{1/4}} = 0 \\ x^{1/4}y^{3/4} = 1 \end{cases}$$

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Divide the first two equations obtaining

$$\begin{cases} 6 = \dfrac{y}{x} \\ x^{1/4}y^{3/4} = 1 \end{cases}$$

Conclusion $x = 6^{-3/4}, \quad y = 6^{1/4}, \quad m = \dfrac{4 \times 2^{1/4}}{3^{3/4}}$

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Bordered Hessian

$$\Lambda = \begin{bmatrix} \dfrac{3my^{3/4}}{16x^{7/4}} & -\dfrac{3m}{16x^{3/4}y^{1/4}} & \dfrac{y^{3/4}}{4x^{3/4}} \\ -\dfrac{3m}{16x^{3/4}y^{1/4}} & \dfrac{3mx^{1/4}}{16y^{5/4}} & \dfrac{3x^{1/4}}{4y^{1/4}} \\ \dfrac{y^{3/4}}{4x^{3/4}} & \dfrac{3x^{1/4}}{4y^{1/4}} & 0 \end{bmatrix}$$

Substituting critical point

$$\Lambda = \begin{bmatrix} \frac{3\times 3^{3/4}}{\sqrt[4]{2}} & -\frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{3^{3/4}}{2\sqrt[4]{2}} \\ -\frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{1}{4\sqrt[4]{6}} & \frac{3^{3/4}}{4\sqrt[4]{2}} \\ \frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{3^{3/4}}{4\sqrt[4]{2}} & 0 \end{bmatrix}$$

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Then $\det \Lambda = -\dfrac{3\sqrt[4]{3}}{2^{3/4}}$ thus we have a minimum

Esempio

Minimizzare la funzione $f(x,y) = x^2 + 2y^2 + 3z^2 + 2xz + 2yz$ **sottoposta ai vincoli**

$$x + y + z = 1 \qquad 2x + y + 3z = 7$$

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la Lagrangiana è

$$L(x, y, z; m, n) = x^2 + 2y^2 + 3z^2 + 2xz + 2yz - m(x + y + z - 1) - n(2x + y + 3z - 7)$$

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Condizioni di ottimalità

$$\begin{cases} 2x + 2z = m + 2n \\ 4y + 2z = m + n \\ 2x + 2y + 6z = m + 3n \\ x + y + z = 1 \\ 2x + y + 3z = 7 \end{cases}$$

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$x = 0,\ y = -2,\ z = 3,\ m = -10,\ n = 8$

Ordinary Differential Equations

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For instance let $y = y(x)$ be some function of the independent variable x. The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) := y'(x) = 2xy(x) \tag{1}$$

states that the first derivative of the function y equals the product of $2x$ and the function y itself. An additional, implicit statement in this differential equation is that the stated relationship holds only for all x for which both the function and its first derivative are defined

Given $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

$$\begin{cases} y'(x) = f(x, y(x)), & x \in I \\ y(x_0) = y_0, & x_0 \in I,\ y_0 \in J \end{cases}$$

where $I \times J \subseteq \Omega$ is called initial value problem We say that differential equations are studied by quantitative or exact methods when they can be solved completely, i.e. all the solutions are known and could be written in closed form in terms of elementary functions or sometime special functions (or inverses of these type of functions).

Existence: Peano's Theorem

If $f(x, y)$ is continuous on $\mathcal{R} = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$, then the initial value problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

has a solution in a neighborhood of x_0

Solution needs not to be unique.

Consider the initial value problem

$$\begin{cases} y'(x) = 2\sqrt{|y(x)|} \\ y(0) = 0 \end{cases} \tag{p}$$

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$y(x) = 0$ solves (p)

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$$\begin{cases} y'(x) = 2\sqrt{|y(x)|} \\ y(0) = 0 \end{cases} \tag{p}$$

$y(x) = 0$ solves (p)

$y(x) = x|x|$ solves (p)

Moreover for each pair of real numbers $\alpha < 0 < \beta$

$$\varphi_{\alpha,\beta}(x) = \begin{cases} -(x-\alpha)^2 & \text{if} \quad x < \alpha \\ 0 & \text{if} \quad \alpha \le x \le \beta \\ (x-\beta)^2 & \text{if} \quad x > \beta \end{cases}$$

solves (p)

Existence and uniqueness: Picard Lindelhöf Theorem Let the function $f(x, y)$ continuous on a rectangle $\mathcal{R} = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. Suppose, furthermore, that for any y_1, $y_2 \in [y_0 - b, y_0 + b]$ there exists $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2| \tag{L}$$

for each $x \in [a, b]$. Then the initial value problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \tag{ivp}$$

has a unique solution defined in $[x_0 - \delta, x_0 + \delta]$ where $\delta = \min\{a,\ b/M\}$ being $M := \max\limits_{(x,y)\in\mathcal{R}} |f(x, y)|$

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In fact assume $|f_y(x, y)| = \left|\frac{\partial f}{\partial y}(x, y)\right| \leq L$ then from the mean value (Lagrange) theorem

$$|f(x, y_1) - f(x, y_2)| = |f_y(x, y)|\,|y_1 - y_2|$$

with y between y_1 and y_2